Hankel-Based Methods for Latent-State Structured Prediction

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LxMLS 2014

Structured Prediction

- We want to learn functions over structured objects:
  - from sequences to reals (e.g. sequence classification)
  - from pairs of sequences to reals (e.g. sequence tagging)
  - from pairs of sequences and trees to reals (e.g. NLP parsing)
- Stochastic Functions: $0 \leqslant f(x) \leqslant 1$ and $\sum_{x \in X} f(x) = 1$
- Use these models to make predictions: $\text{argmax}_x f(x)$

Latent State Models

- Consider a sequence $abaccb$

$$\begin{aligned} f(abaccb) &= \alpha_f(ab) \cdot \beta_f(accb) \\ &= \alpha_f(ab) \cdot A_a \cdot \beta_f(ccb) \\ &= \alpha_f(aba) \cdot A_c \cdot \beta_f(cb) \end{aligned}$$

  where:

  - n is the dimension of the model
  - α_f maps prefixes to $\mathbb{R}^n$
  - β_f maps suffixes to $\mathbb{R}^n$
  - A_a is a bilinear operator in $\mathbb{R}^{n \times n}$

- F : functions computed by WFAs (HMMs is a subclass)

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- F : functions computed by WFAs (HMMs is a subclass)

- Finding $f \in F$: Estimate α_f, β_f and A_σ from S
  - EM
- Many ways of expressing the same $f \in F$
  - Algebraic Methods

Outline

- Spectral Learning of WFA
- Hankel-Based Optimization for Structured Prediction

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- Spectral Learning of WFA → 'Classic spectral learning algorithms'
- Hankel-Based Optimization for Structured Prediction → 'Different interpretation of spectral trick'

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- **Spectral Learning of WFA**
- Hankel-Based Optimization for Structured Prediction

Weighted Automata (WFA)

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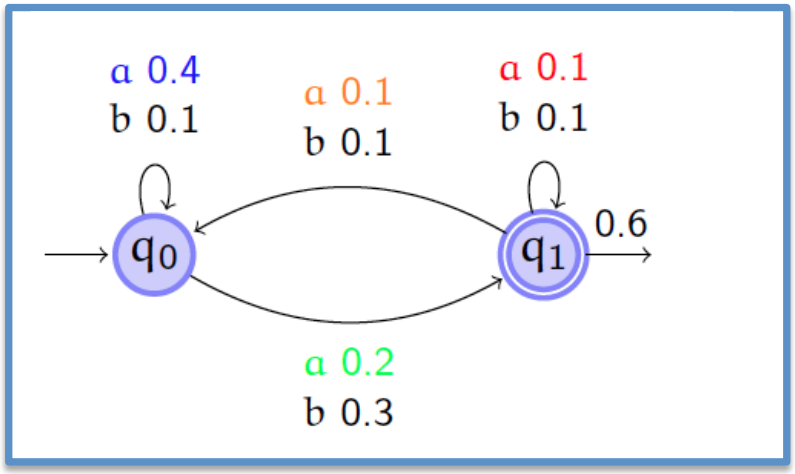

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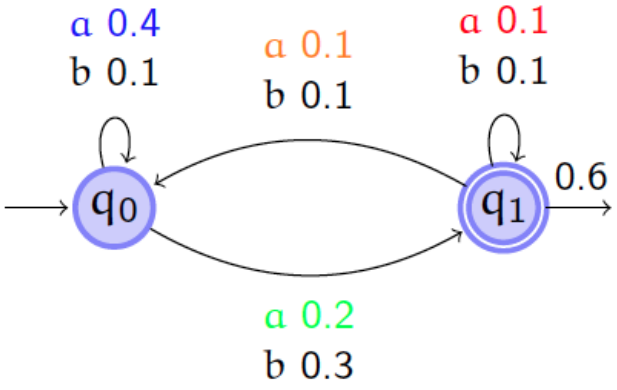

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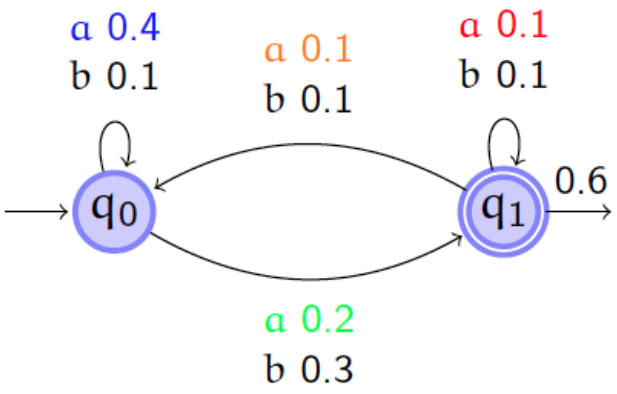

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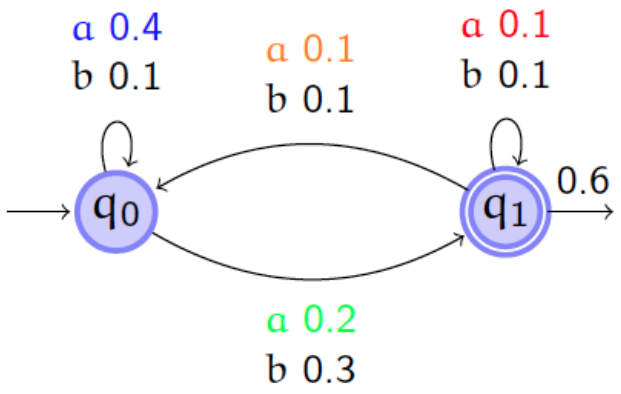

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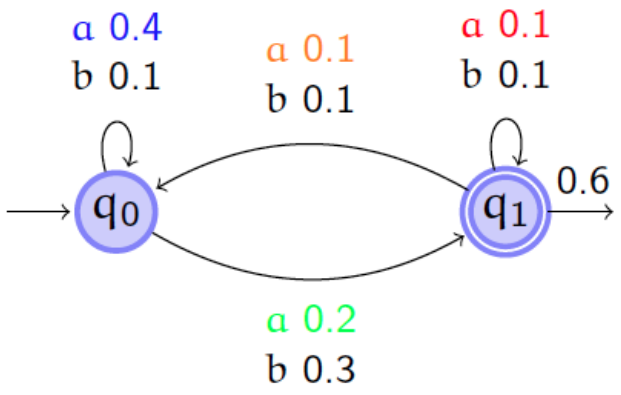

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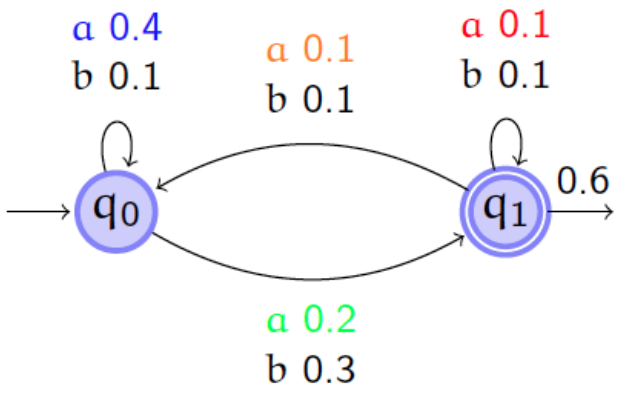

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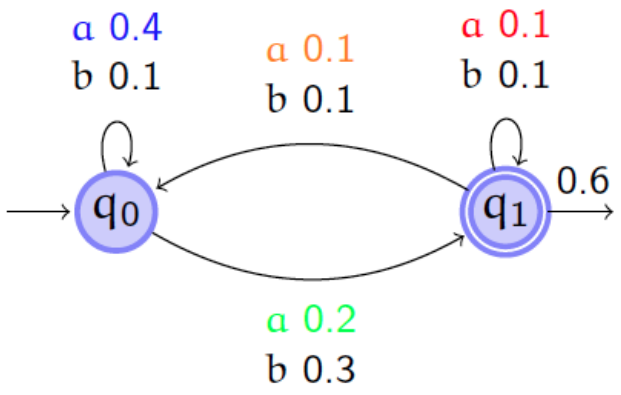

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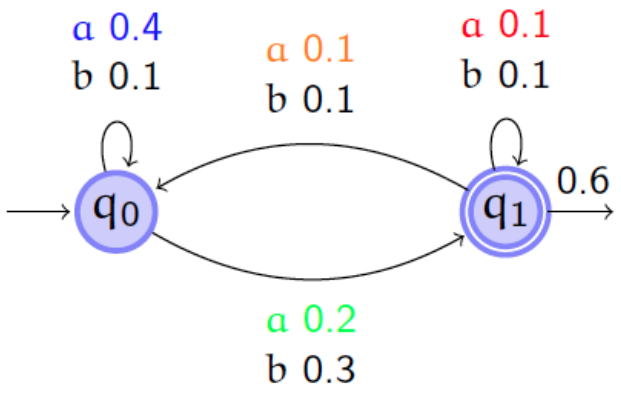

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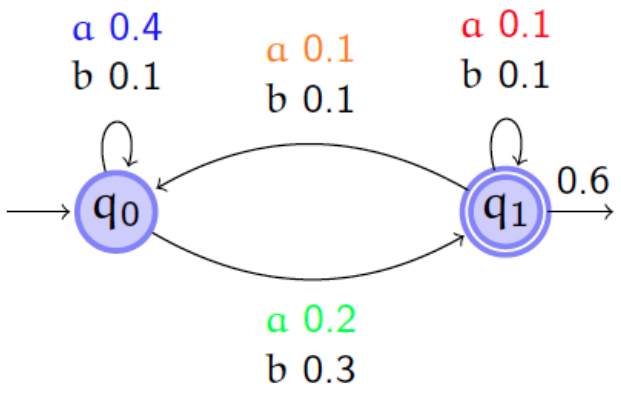

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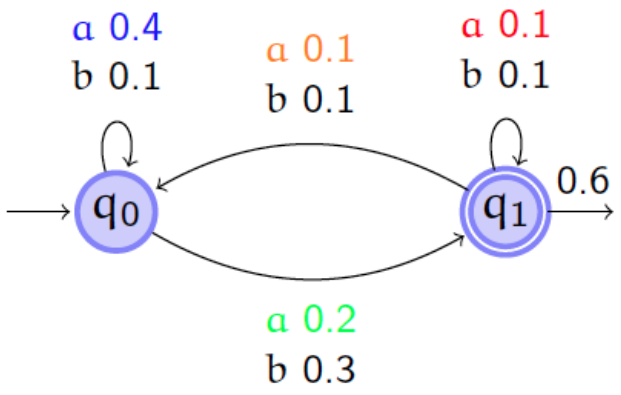

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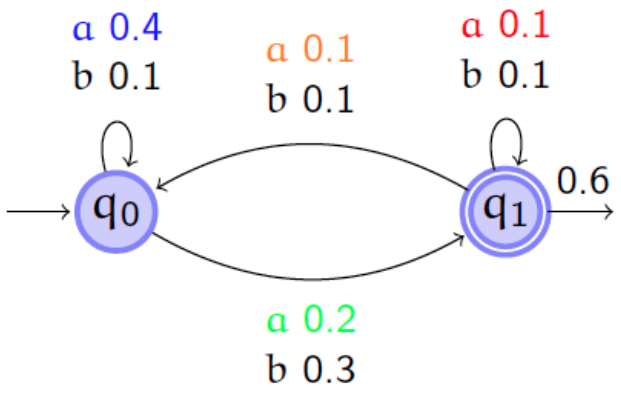

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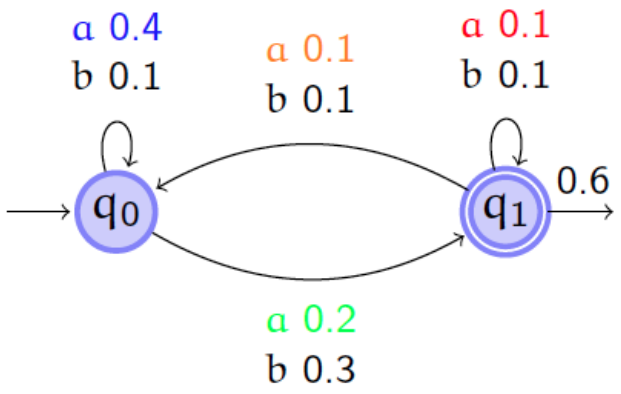

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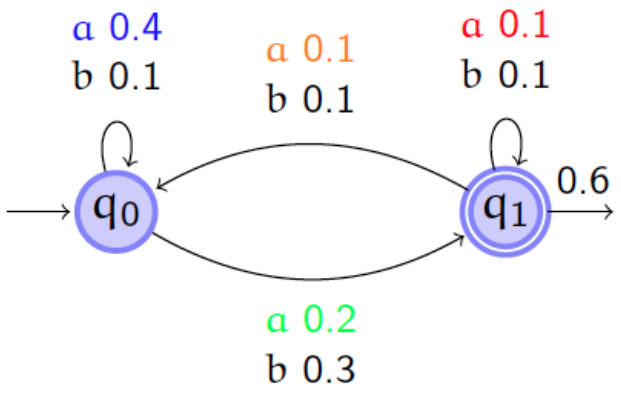

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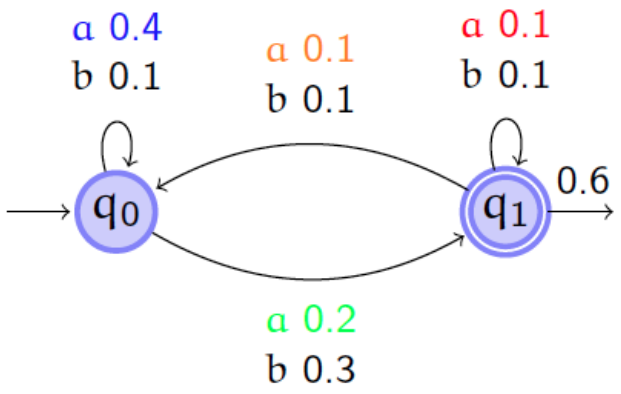

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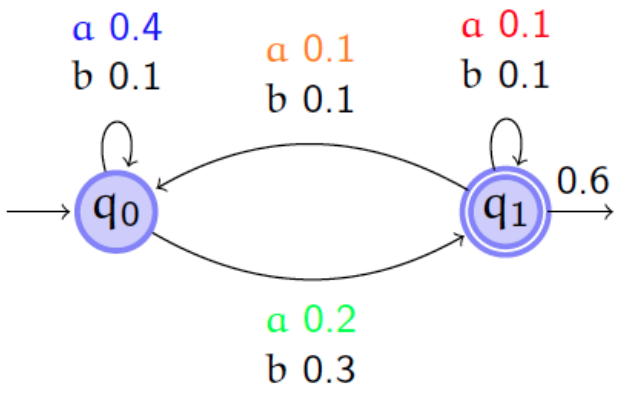

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$$f_{\mathbf{A}}(x) = f_{\mathbf{A}}(x_1 \cdots x_t) = \alpha_1^\top A_{x_1} \cdots A_{x_t} \alpha_\infty$$

Examples of functions computed by WFAs

Generates infinite strings, computes probabilities of prefixes $\mathbb{P}[x\Sigma^\star]$

Emission and transition are conditionally independent given state

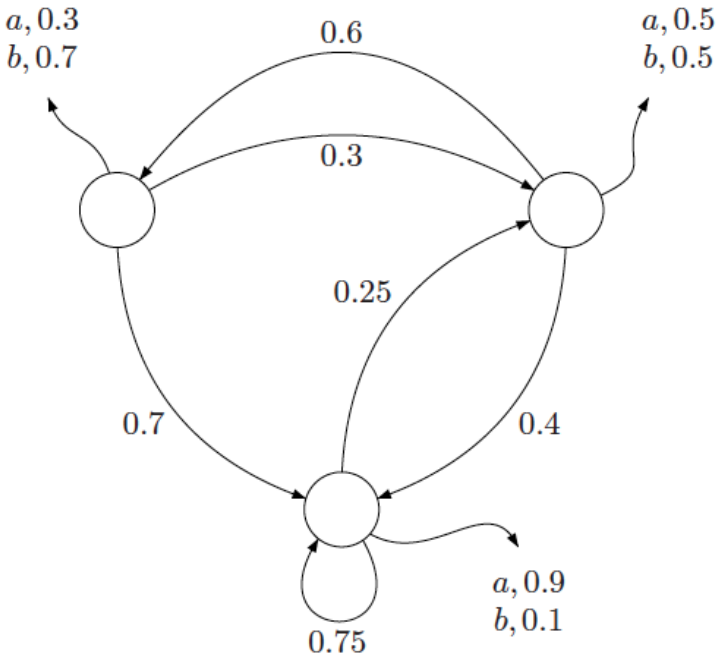
$$\alpha_1^\top = [0.3 \;\; 0.3 \;\; 0.4]$$

$$\alpha_\infty^\top = [1 \;\; 1 \;\; 1]$$

$$A_a = O_a \cdot T$$

$$T = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0 & 0.75 & 0.25 \\ 0 & 0.4 & 0.6 \end{bmatrix}$$

$$O_a = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$



HMMs

Examples of functions computed by WFAs

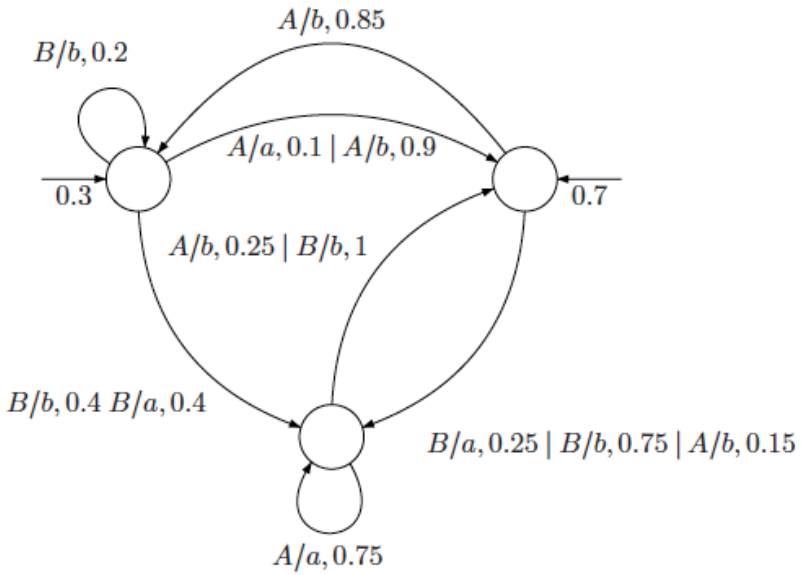
Compute conditional probabilities $\mathbb{P}[y|x] = \alpha_1^\top A_x^y \alpha_\infty$ for pairs $(x, y) \in (\Sigma \times \Delta)^\star$, must have $|x| = |y|$

Can also assume models factorized like in HMM

$$\alpha_1^\top = [0.3 \;\; 0 \;\; 0.7]$$

$$\alpha_\infty^\top = [1 \;\; 1 \;\; 1]$$

$$A_B^b = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0 & 0 & 1 \\ 0 & 0.75 & 0 \end{bmatrix}$$



FSTs

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Consequences:

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Example, for HMMs:

$$[\alpha_{\mathbf{A}}(p)]_i = \mathbb{P}[p\,,\, h_{+1} = i]$$
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Consequences:

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$$
\begin{array}{c}
 \\
\epsilon \\
a \\
b \\
aa \\
ab \\
\vdots
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \epsilon & a & b & aa & ab & \cdots \end{array} \\
\begin{bmatrix}
 & & \vdots & & \vdots & \\
\cdots & \cdots & \vdots & \cdots & f(aab) & \\
 & & \vdots & & & \\
\cdots & \cdots & f(aab) & & & \\
 & & & & & \\
 & & & & &
\end{bmatrix}
\end{array}
$$

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Very *redundant* representation

	ϵ	a	b	aa	ab	$\cdots$
ϵ			$\vdots$		$\vdots$	
a	$\cdots$	$\cdots$	$\vdots$	$\cdots$	$f(aab)$	
b			$\vdots$			
aa	$\cdots$	$\cdots$	$f(aab)$			
ab						
$\vdots$						

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Very *redundant* representation

we will work with finite sub-blocks of H

$$
\begin{array}{c|cccccc}
 & \epsilon & a & b & aa & ab & \cdots \\
\epsilon & & & \vdots & & \vdots & \\
a & \cdots & \cdots & \vdots & \cdots & f(aab) & \\
b & & & \vdots & & & \\
aa & \cdots & \cdots & f(aab) & & & \\
ab & & & & & & \\
\vdots & & & & & &
\end{array}
$$

Hankel Matrix of a WFA

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Schützenberger's Theorem

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Remarks ($\Leftarrow$)

Write $F = \alpha_{\mathbf{A}}(\Sigma^\star)^\top \in \mathbb{R}^{\Sigma^\star \times n}$ and $B = \beta_{\mathbf{A}}(\Sigma^\star) \in \mathbb{R}^{n \times \Sigma^\star}$

Note $H_f = F \cdot B$

Then, $\text{rank}(H_f) = n$

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Then, $\text{rank}(H_f) = n$

Remarks ($\Rightarrow$)

Assume $\text{rank}(H_f) = n$

Take *rank factorization* $H_f = F \cdot B$ with $F \in \mathbb{R}^{\Sigma^\star \times n}$ and $B \in \mathbb{R}^{n \times \Sigma^\star}$

A WA that computes f can be constructed using F and B

Low-rank Hankel and Operators

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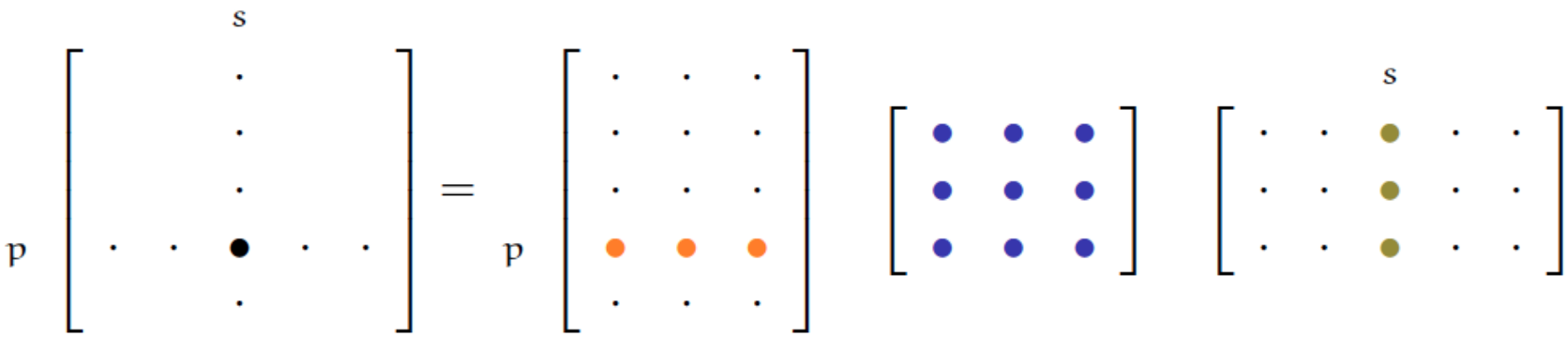
$$H_\sigma \in \mathbb{R}^{\Sigma^\star \times \Sigma^\star} \qquad F \in \mathbb{R}^{\Sigma^\star \times n} \qquad A_\sigma \in \mathbb{R}^{n \times n} \qquad B \in \mathbb{R}^{n \times \Sigma^\star}$$



$$f(p_1 \cdots p_t \cdot \sigma \cdot s_1 \cdots s_{t'}) = \underbrace{\alpha_1^\top A_{p_1} \cdots A_{p_t}}_{\alpha(p)^\top} \; A_\sigma \; \underbrace{A_{s_1} \cdots A_{s_{t'}} \alpha_\infty}_{\beta(s)}$$

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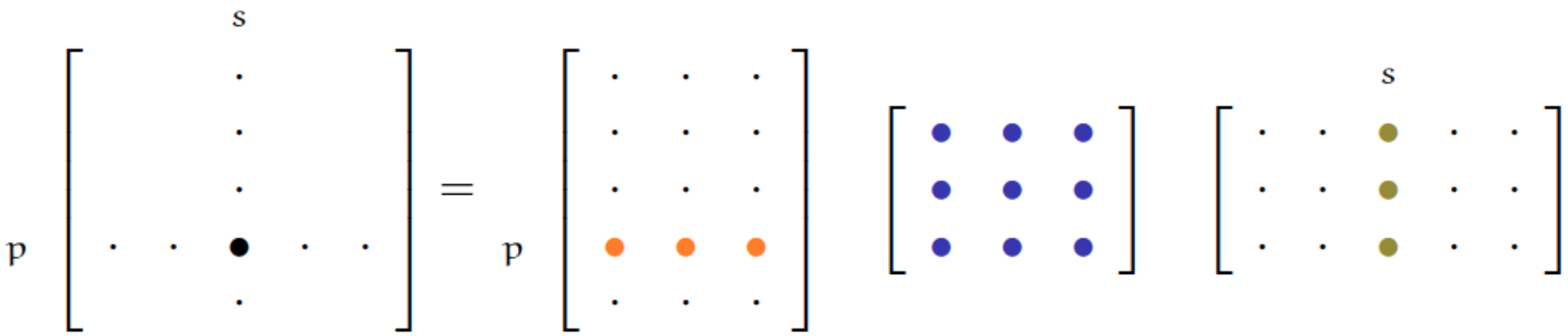

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$$H_\sigma = F\, A_\sigma\, B \quad \Longrightarrow \quad A_\sigma = F^+\, H_\sigma\, B^+$$

Spectral Method

Spectral Method

Idea: Use SVD decomposition to obtain a factorization of H

- Choose a basis: sets of prefixes $\mathcal{P}$ and suffixes $\mathcal{S}$
- Obtain H and H_σ over basis $(\mathcal{P}, \mathcal{S})$
- Compute *compact* SVD as $H = U\Lambda V^\top$ with

$$U \in \mathbb{R}^{\mathcal{P} \times n} \qquad \Lambda \in \mathbb{R}^{n \times n} \qquad V \in \mathbb{R}^{\mathcal{S} \times n}$$

- Construct the WA using $F = U\Lambda$ and $B = V^\top$:
  - $A_\sigma = F^+ H_\sigma V^+$ for all $\sigma \in \Sigma$
  - $\alpha_1^\top = H(\lambda, \cdot)^\top B^+, \qquad \alpha_\infty = F^+ H(\cdot, \lambda)$

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Properties

- Easy to implement: just linear algebra
- Fast to compute: $O(\max\{|\mathcal{P}|, |\mathcal{S}|\}^3)$
- Noise tolerant: $\hat{H} \approx H$ and $\hat{H}_\sigma \approx H_\sigma$ implies $\hat{A}_\sigma \approx A_\sigma$

Spectral Method

Choice of Preffix-Suffix Basis $(\mathcal{P}, \mathcal{S})$

- Should satisfy that $\text{rank}(H_{\Sigma^\star \times \Sigma^\star}) = \text{rank}(H_{\mathcal{P} \times \mathcal{S}})$
  - $\mathcal{P} = \mathcal{S} = \Sigma \cup \{\lambda\}$ (Hsu et al COLT-09)
  - all prefixes/suffixes in the training data
  - most frequent ones (Balle et al MLJ-13)
  - feature-based equivalence classes (Cohen et al NAACL-13)

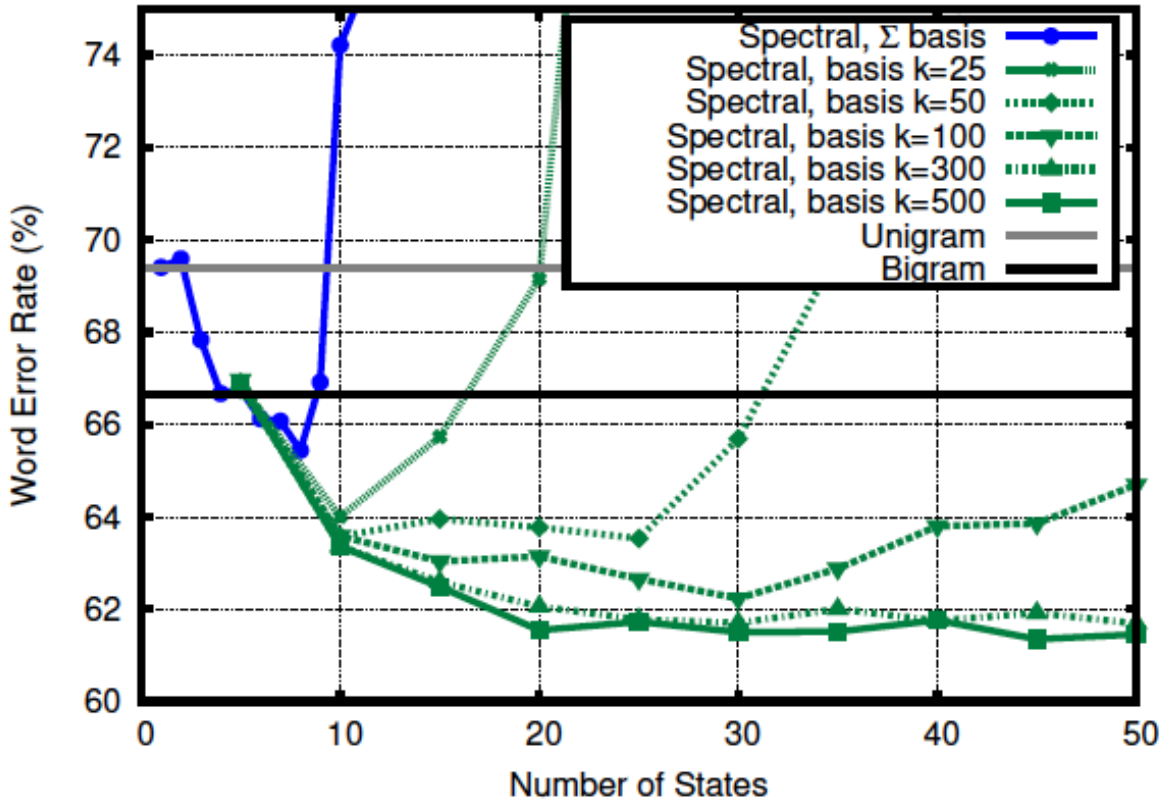
Observable Statistics in H and H_σ

- The WA can be recovered from
  - String probabilities $\mathbb{P}[x]$
  - Prefix probabilities $\mathbb{P}[x\Sigma^*]$ (Hsu et al COLT-09)
  - Substring expectations $\mathbb{E}[|w|_x]$
- See (Balle et al MLJ-13) for details

Parameter of the method

- A dimension n: controls complexity of the WA family

Experiment: PoS-Tag Sequence Models

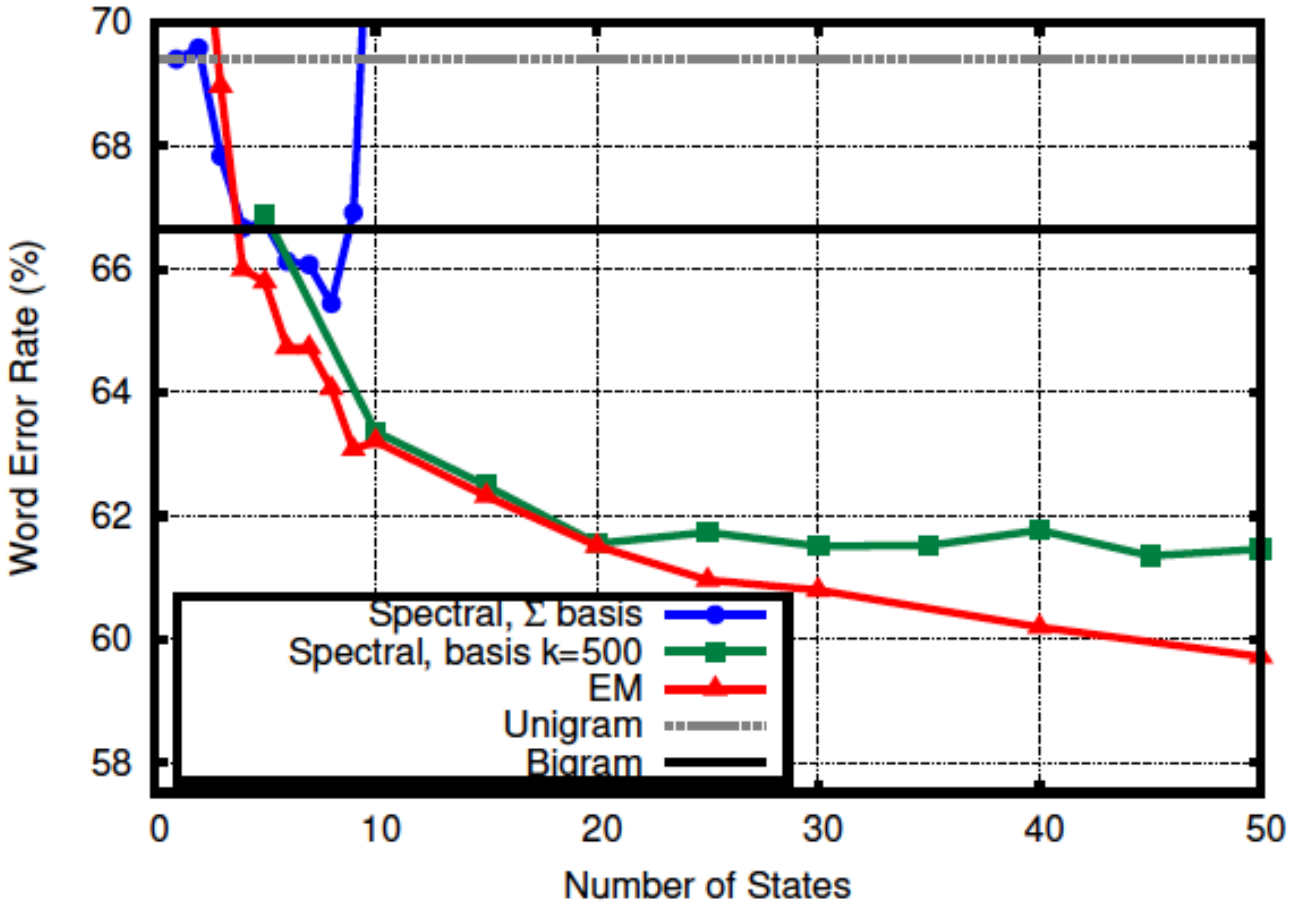


PTB sequences of simplified PoS tags (Petrov et al 2010)

Configuration: expectations on frequent substrings

Metric: error rate on predicting next symbol in test sequences

Experiment: PoS-Tag Sequence Models



Comparison with a bigram baseline and EM

Metric: error rate on predicting next symbol in test sequences

At training, the Spectral Method is > 100 faster than EM

Classical Spectral: Summary

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Learning Setting:

- Considers stochastic functions
- Assumes there is a model that generates sequences
- We can sample from this model

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Unknown PWFA

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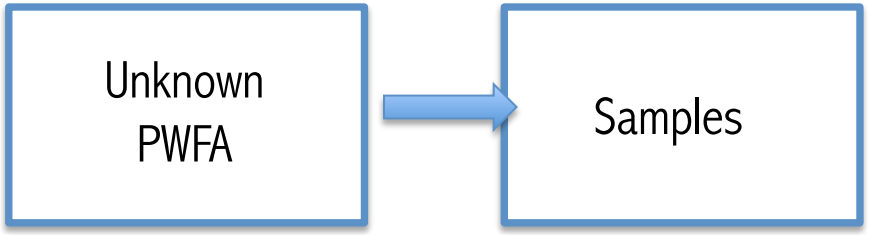

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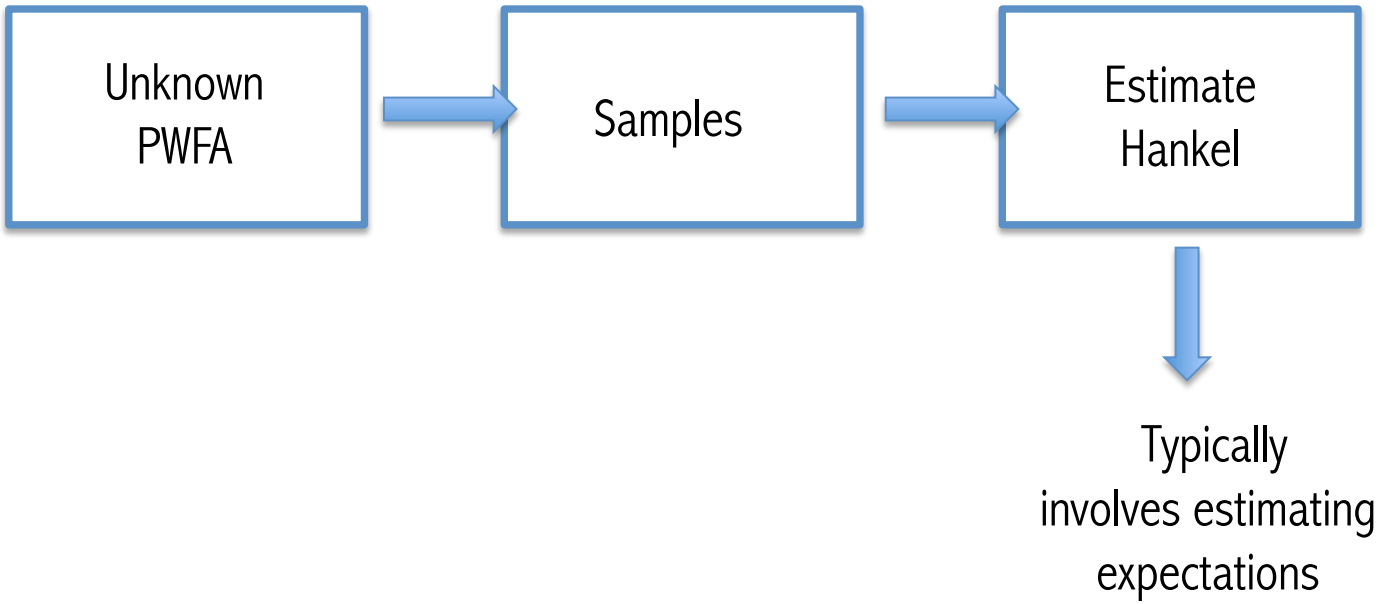

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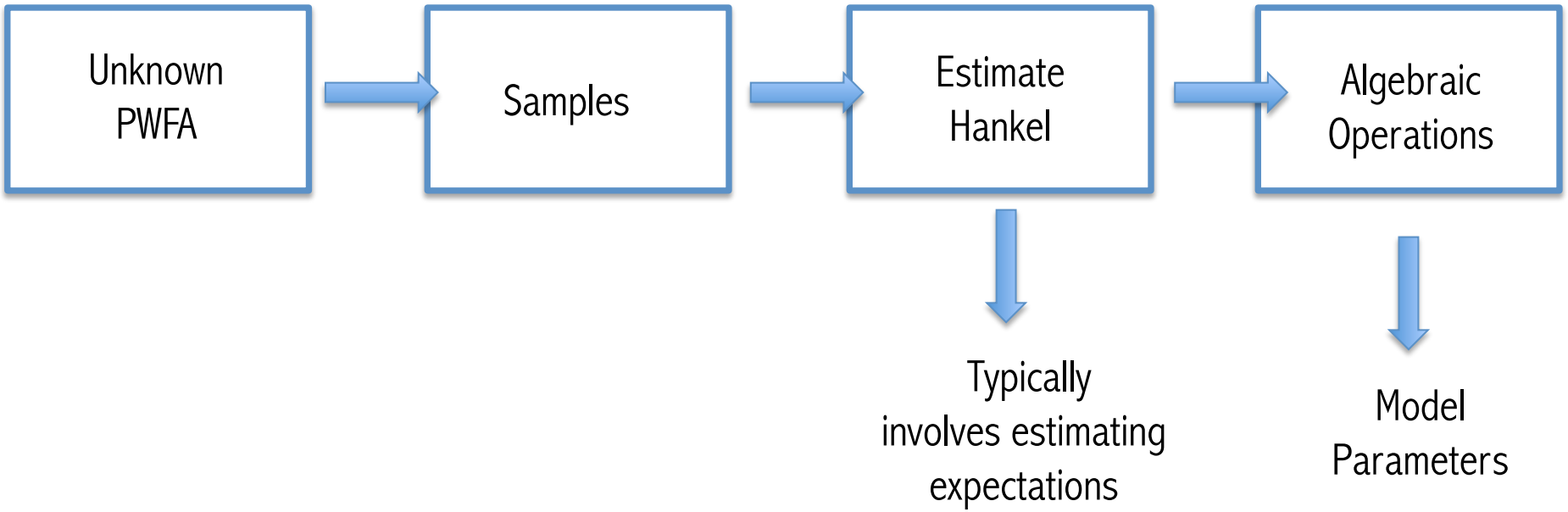

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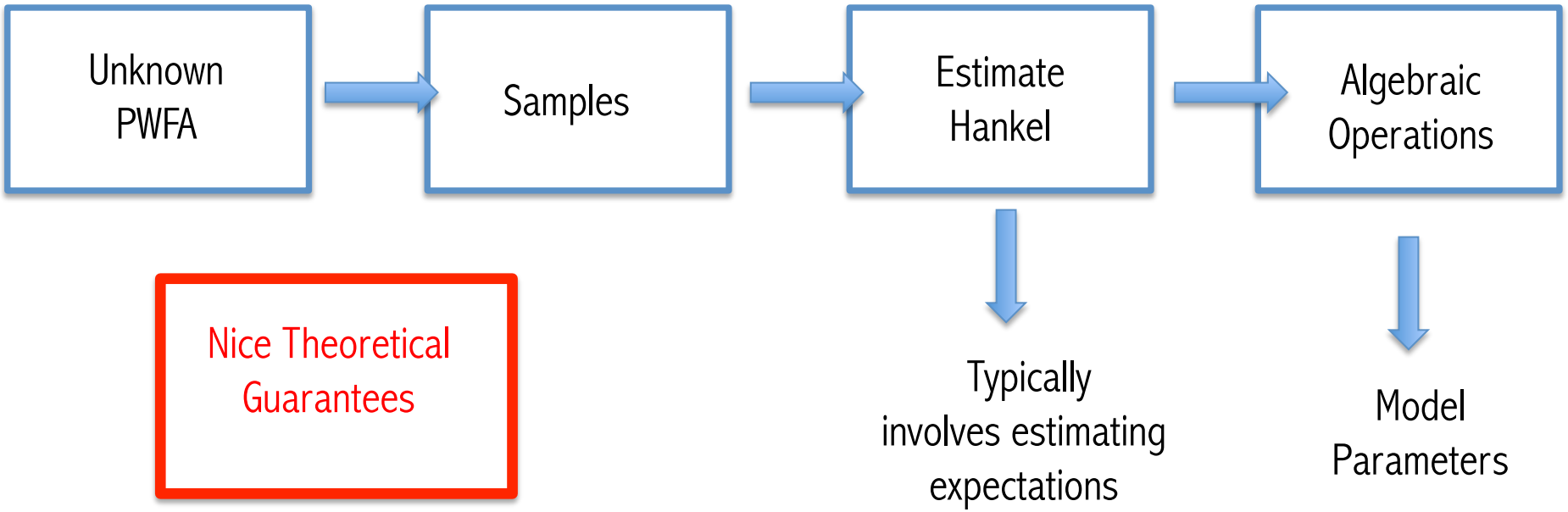

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- Necessary statistics are hard to estimate
- In some cases we can not directly estimate the Hankel from samples

  eg. Learning a function that is not a distribution

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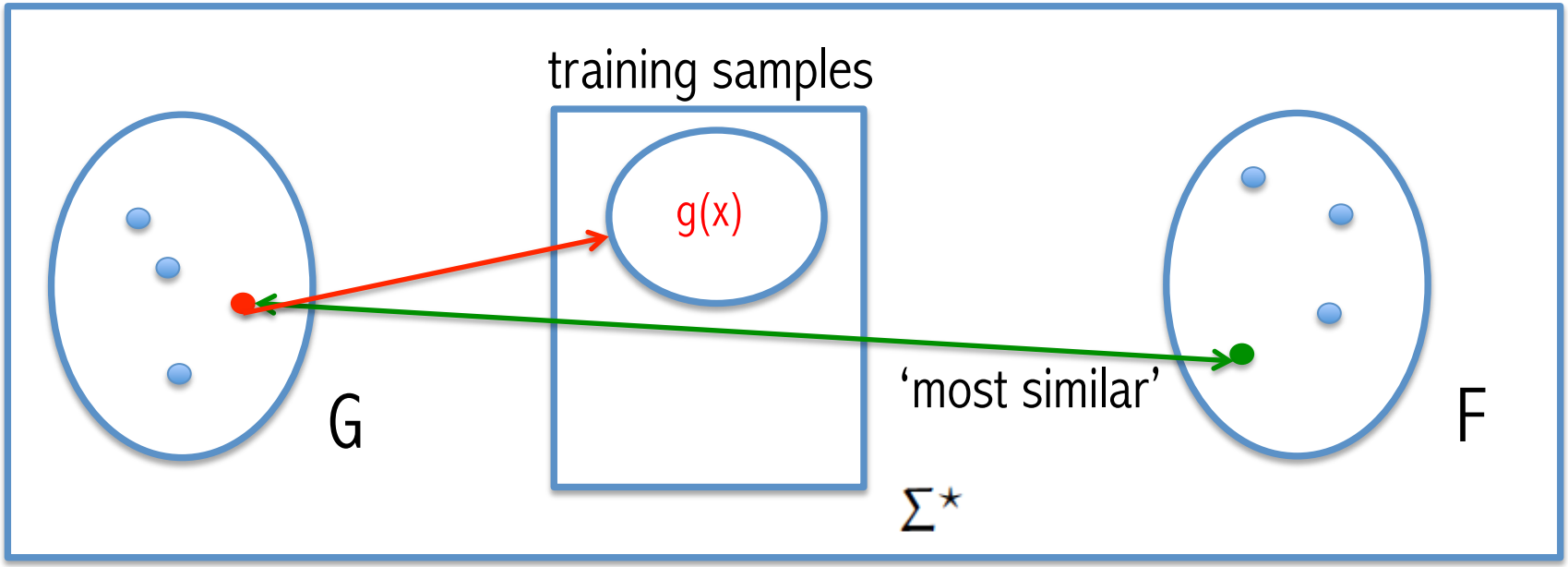

Outline

- Spectral Learning of WFA
- **Hankel-Based Optimization for Structured Prediction**

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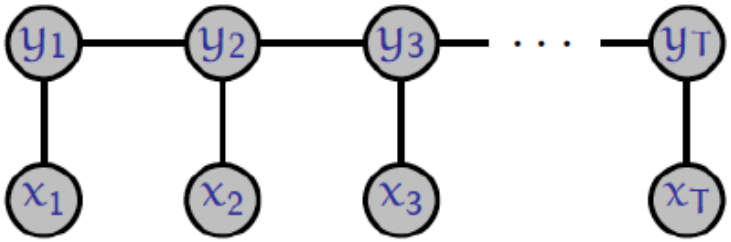
- Spectral Learning of WFA
- **Hankel-Based Optimization for Structured Prediction** → Sequence Tagging

Sequence Tagging

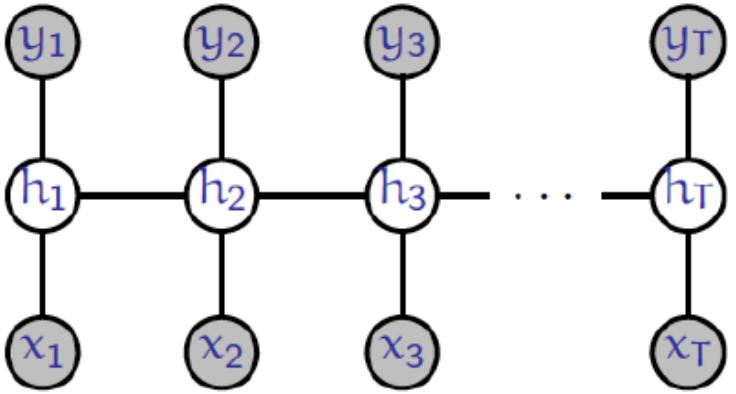
output:	h	l	p	-	x	p	a	t	x	m	x	s
input:	h	i	p	p	o	p	o	t	a	m	u	s

Fully Observable Models



+ Making predictions is tractable
+ Learning is convex
– Performance crucially depends on features

Latent-variable Models



+ Hidden layer provides more expressivity
– Making predicitons is not tractable
– Learning is non-convex (this paper)

Learning Structured Predictors with Latent Variables

Desiderata:

- Expressive scoring functions
- Tractable prediction function
- Effective regularizer
- *Convex training procedure*

Main Idea: Change of Representation + Relaxation

- Problem Formulation
  - Scoring functions computed by Input-Output OOMs
  - Piecewise Prediction and Loss Function

- Solving the Learning Problem
  - Spectral trick:
    optimize over parameters of $f \rightarrow$ optimize low-rank matrix H
  - Relax low-rank constraint using nuclear norm of H
  - Recover parameters of f from H using the spectral method

Scoring Functions Computed by IO-OOM

Latent Score $\theta(x, y, h)$:

$$\alpha(h_0) \prod_{t=1}^{T} A_{y_t}^{x_t}(h_{t-1}, h_t)\ \beta(h_T)$$

Scoring Function $F_A(x, y)$:

$$\sum_h \theta(x, y, h) = \alpha^T\ A_{y_1}^{x_1} \dots A_{y_T}^{x_T}\ \beta$$

- Model: $A : \langle \alpha, \beta, \{A_b^a\} \rangle$
- Number of states: n
- Initital Weights: $\alpha \in \mathbb{R}^n$
- Final Weights: $\beta \in \mathbb{R}^n$
- Observable Operators $A_b^a \in \mathbb{R}^{n \times n}$

- Expressive Function Family $\rightarrow$ e.g. it includes HMM
- Making Predictions (i.e. maximizing $F_A(x, y)$) $\rightarrow$ NP-hard

Piecewise Prediction and Loss for IO-OOM

Approximation: $F_A^k(x, y)$:

$$\sum_{t=1}^{T-(k-1)} F_A(x_{t:t+k-1}, y_{t:t+k-1})$$

- Factor size: k
- Sum k–grams
- Task loss: $l(y, z)$
  e.g. hamming distance

Loss $L_k(x, y, F_A)$:

$$\max_z [F_A^k(x, z) - F_A^k(x, y) + l(y, z))$$

- Prediction and Loss Function $\rightarrow$ computed in $O(T|Y|^k)$
  using the Viterbi Algorithm

Discrete Regularizer for IO-OOM

Learning Problem:

$$\underset{A \in \mathcal{F}}{\operatorname{argmin}} \sum_{i=1}^{m} L_k(x^i, y^i, F_A) + \tau |A|)$$

- Function class (IO-OOM): $\mathcal{F}$
- Training Examples: $\langle x^i, y^i \rangle$
- Loss Function: L_k
- Regularizer $\rightarrow$ number of states: $|A|$
- Trade-off constant: τ

- $k \geqslant 2 \rightarrow$ Non-convex dependence of L_k on parameters of A
- L_k involves polynomials of order $k + 3$

Optimization Strategy

- L_k convex on values of A $\rightarrow$ optimization over $(X \times Y)^k$ values
- Three challenges
  1. Table of values $\rightarrow$ must correspond to valid IO-OOM
  2. Regularizer over table $\rightarrow$ must correspond to #states of IO-OOM
  3. Recover parameters of A from this table

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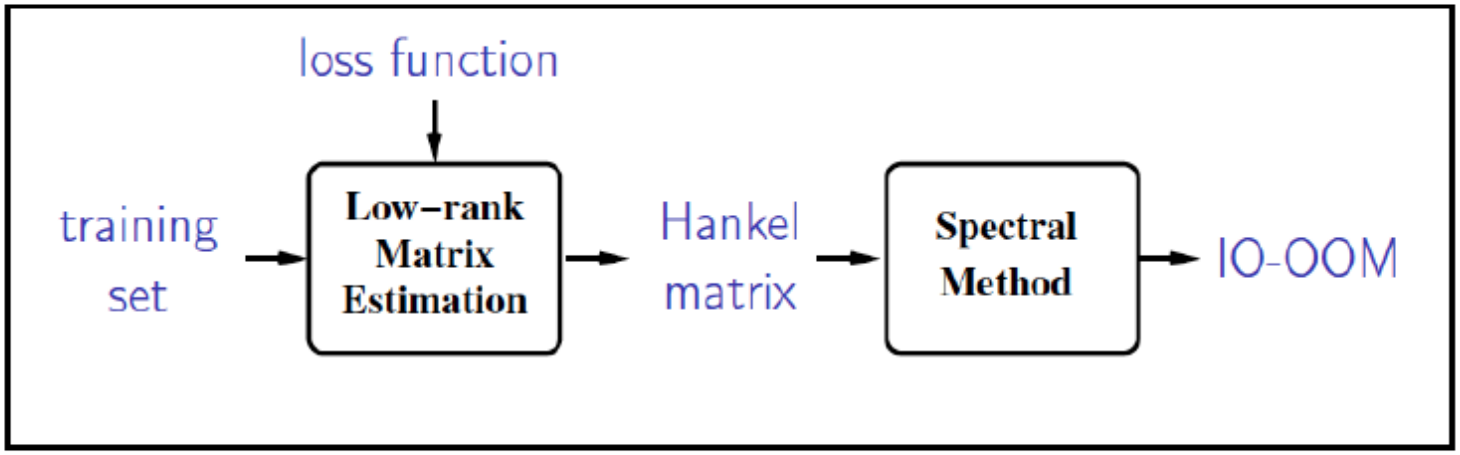
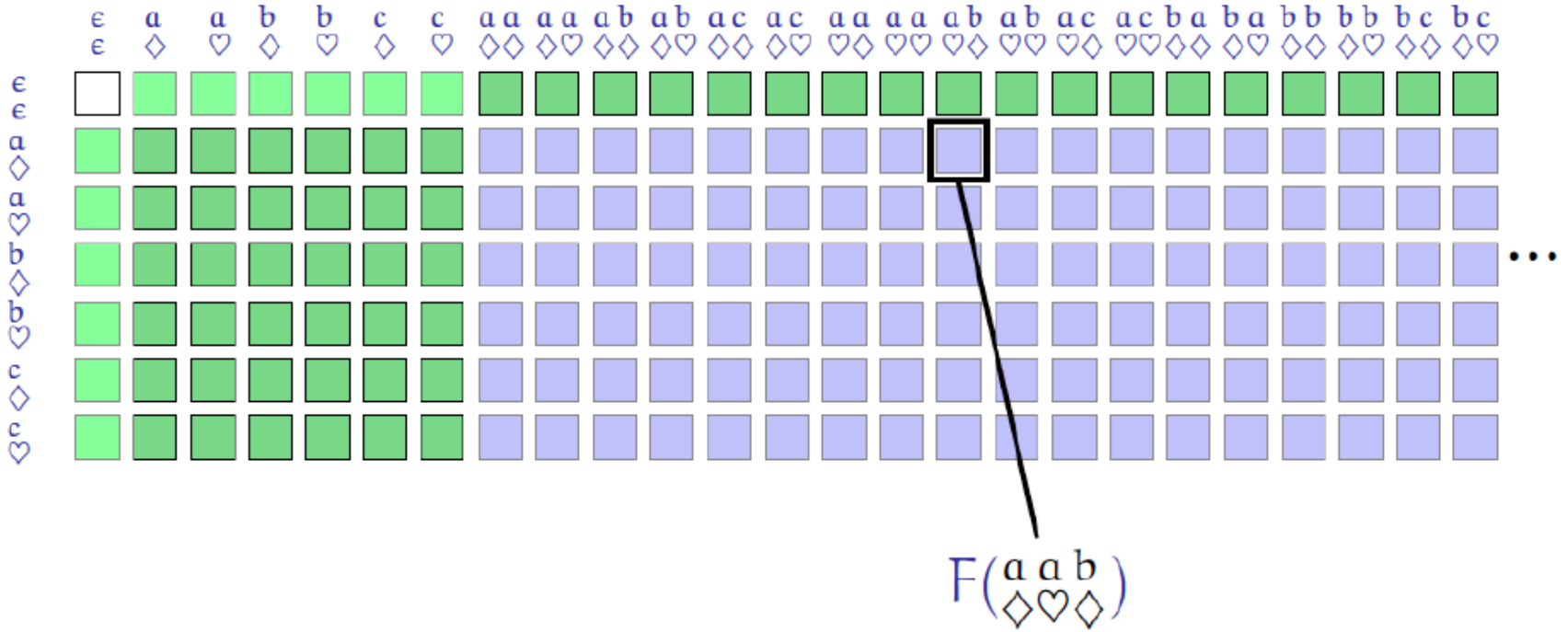
Solution: the Spectral Trick



IO-OOM and Hankel Matrices

$$X = \{a, b, c\} \quad Y = \{\diamondsuit, \heartsuit\}$$



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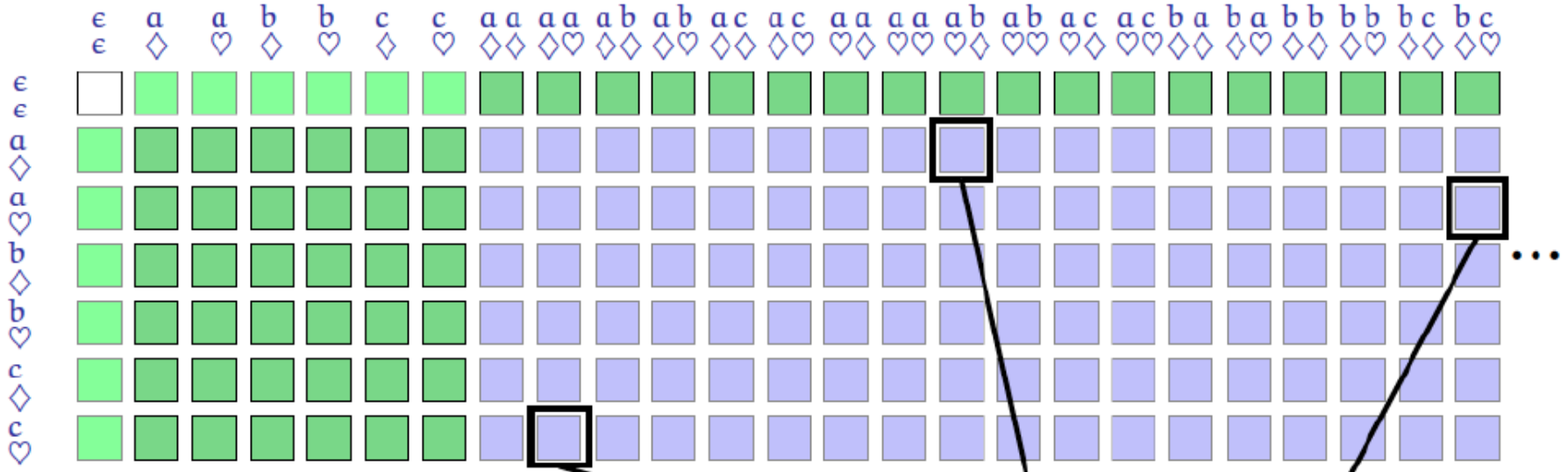

$$F_{k=3}\left(\begin{matrix} c & a & a & b & c \\ \heartsuit & \diamondsuit & \heartsuit & \diamondsuit & \heartsuit \end{matrix}\right) = F\left(\begin{matrix} c & a & a \\ \heartsuit & \diamondsuit & \heartsuit \end{matrix}\right) + F\left(\begin{matrix} a & a & b \\ \diamondsuit & \heartsuit & \diamondsuit \end{matrix}\right) + F\left(\begin{matrix} a & b & c \\ \heartsuit & \diamondsuit & \heartsuit \end{matrix}\right)$$

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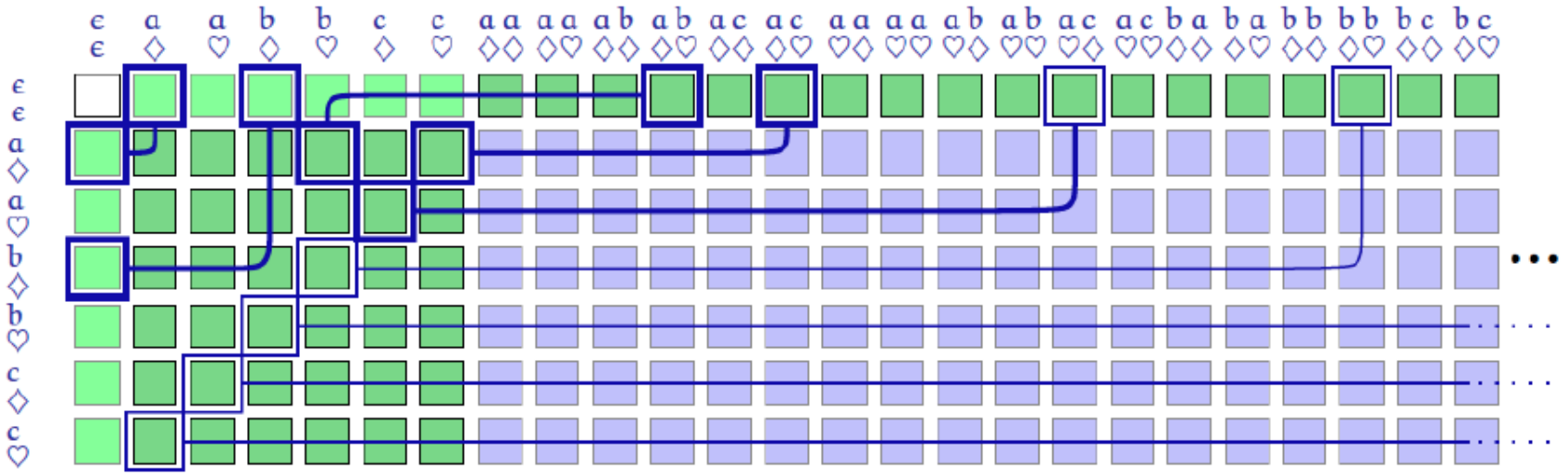


Hankel Structure:

- Equality constraints
- Low-rank constraints

Fundamental Theorem:

F is realized by an n-state IO-OOM

$\Longleftrightarrow$

H has rank at most n for every basis

Max-Margin Completion of Hankel Matrices

Optimization with rank regularization:

$$\underset{H \in \mathbb{H}(P,S)}{\operatorname{argmin}} \sum_{i=1}^{m} L_k(x^i, y^i, H) + \tau \operatorname{rank}(H)$$

Convex relaxation:

$$\underset{H \in \mathbb{H}(P,S)}{\operatorname{argmin}} \sum_{i=1}^{m} L_k(x^i, y^i, H) + \tau ||H||_*$$

- Set of Hankel Matrices over some basis: $\mathbb{H}(P, S)$
- Rank regularizer: $\operatorname{rank}(H)$
- Nuclear norm relaxation: $||H||_*$

- Optimization almost equivalent $\rightarrow$ we search over IO-OOM that can be recovered from $H \in \mathbb{H}(P, S)$
- Once we solve for H we can recover parameters using the spectral technique

Estimation of Hankel Matrices via Convex Optimization

FOBOS Algorithm: Minimization of $L(H) + \tau ||H||_*$

- Initialize: $H_0 = 0$
- while $t \leqslant MaxIter$ do
  - Set G_t to a subgradient of $L(H)$ at H_t
  - Set $H_{t+0.5} = H_t - \frac{c}{\sqrt{t}} G_t$
  - Calculate the SVD of $H_{t+0.5} = U\Sigma V^\top$
  - Define a diagonal matrix Σ' such that $\sigma_i^{'} = max[\sigma_i - \nu_t \tau, 0]$
  - set $H_{t+1} = U\Sigma' V^\top$

  end while

Spectral Recovery

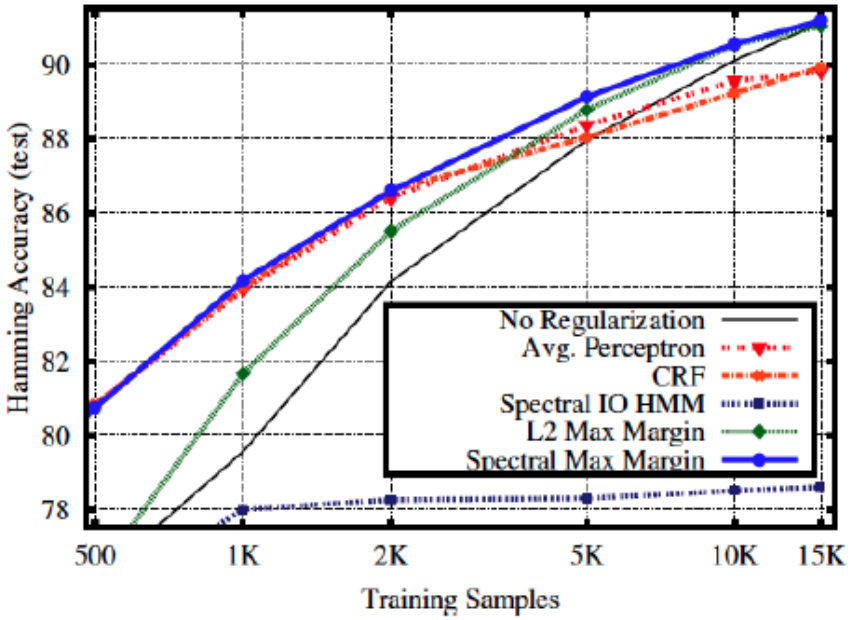
using the method by (Hsu et al. 2009)

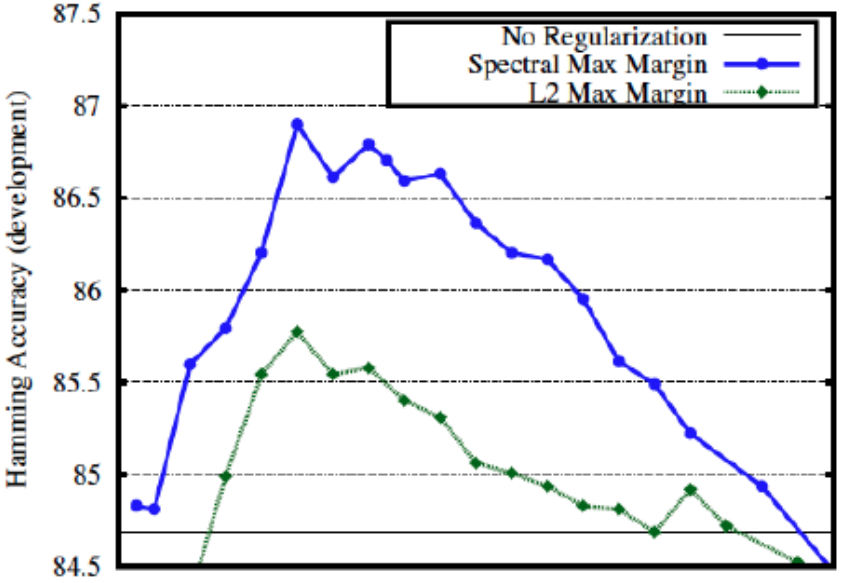
- Spectral Algorithm for IO-OOM
  - Assume F is realized by a minimal n-state IO-OOM A
  - We are given a basis (P, S) such that H has rank n
  - We are given corresponding H_b^a
  - To recover parameters of A:
    - Perform SVD to get $H = U\Sigma V^\top$
    - Define $A_b^a = (HV)^+ H_b^a V$
- Typical spectral algorithms assume that we can estimate H
- In contrast, we regard H as an optimization variable in a loss minimization procedure

Experiments

- Task: Phonetic Transcription (UCI "Nettalk" Dataset)

@	p	-	L	-		h	I	p	-	x	p	a	t	x	m	x	s
a	p	p	I	e		h	i	p	p	o	p	o	t	a	m	u	s





New Regularization for Factorized Linear Models

New Regularization for Factorized Linear Models

Feature Function

$$\phi_{abc;ABC}(x, y, t) \Rightarrow \begin{array}{ll} 1 & \text{if } \begin{array}{l} y_{t-1}y_t y_{t+1} = ABC \\ x_{t-1}x_t x_{t+1} = abc \end{array} \\ 0 & \text{otherwise} \end{array}$$

feature vector

$$\phi(x, y, t) \in \mathbb{R}^{|X|^3 \times |Y|^3}$$

parameter vector

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Specifically designed for structured prediction

Spectral Trick:
Change of Representation

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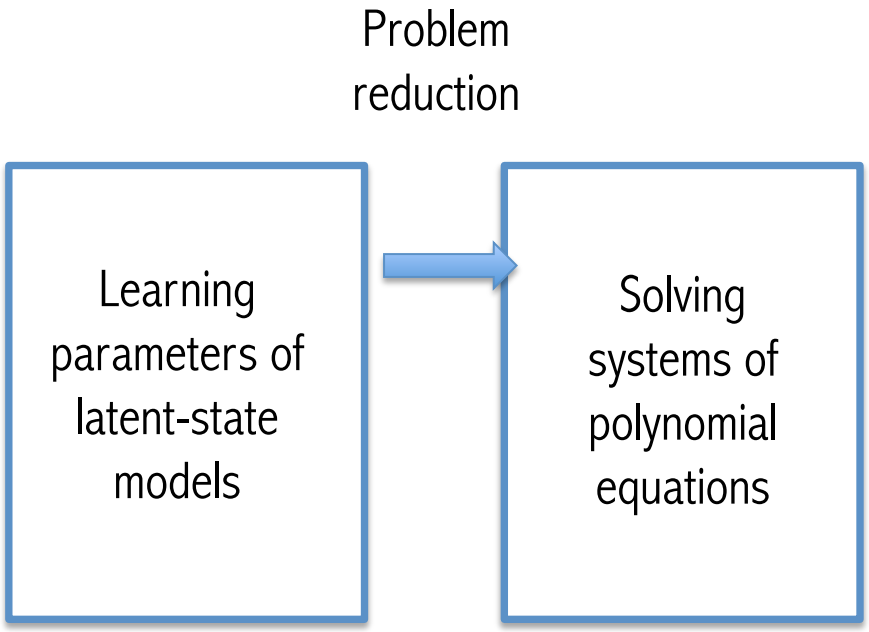
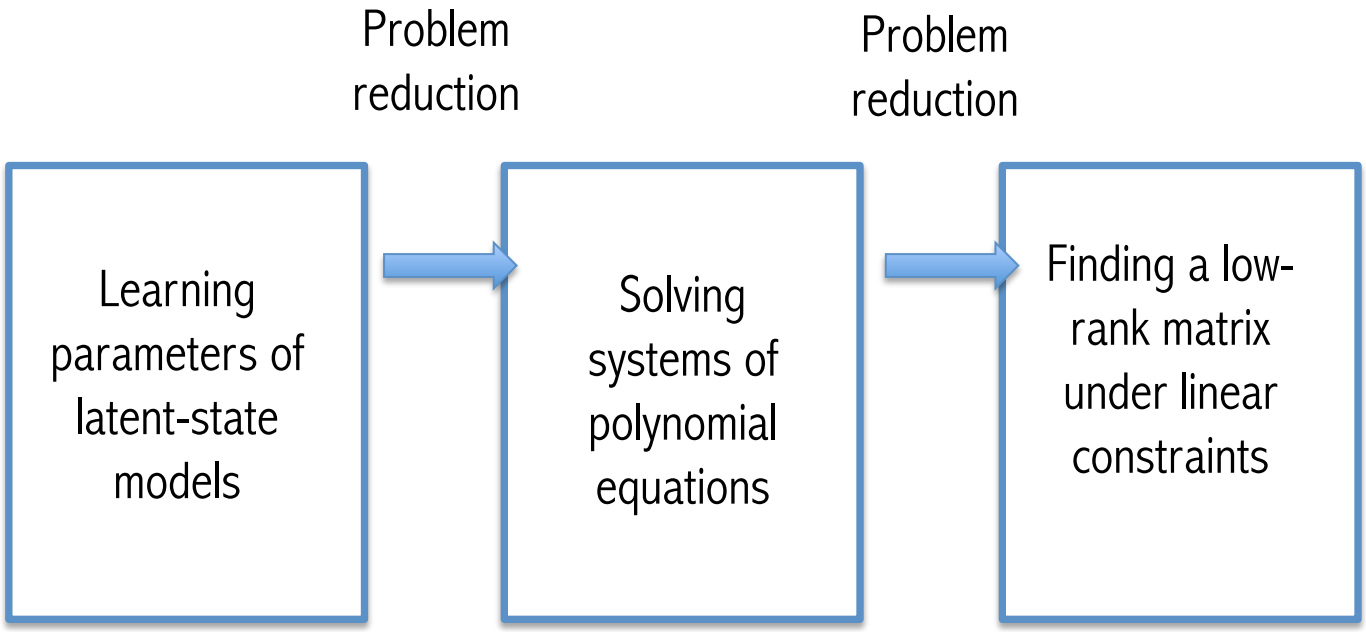
Learning parameters of latent-state models

Spectral Trick: Change of Representation



Spectral Trick: Change of Representation



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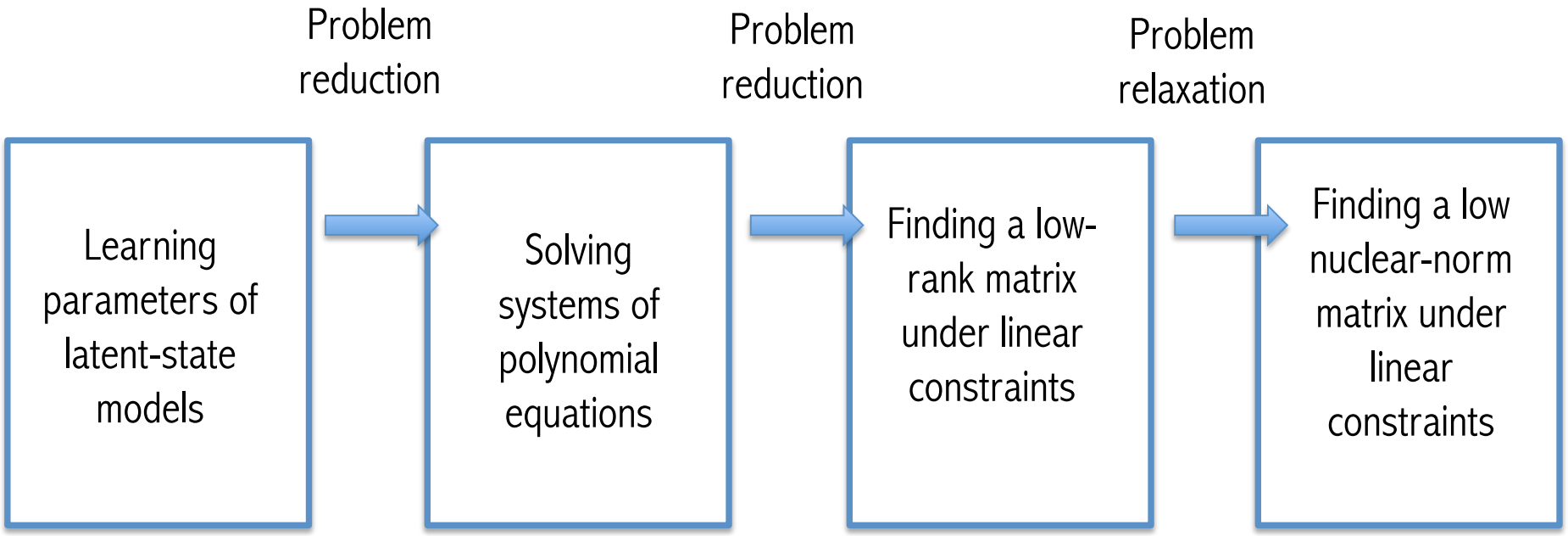

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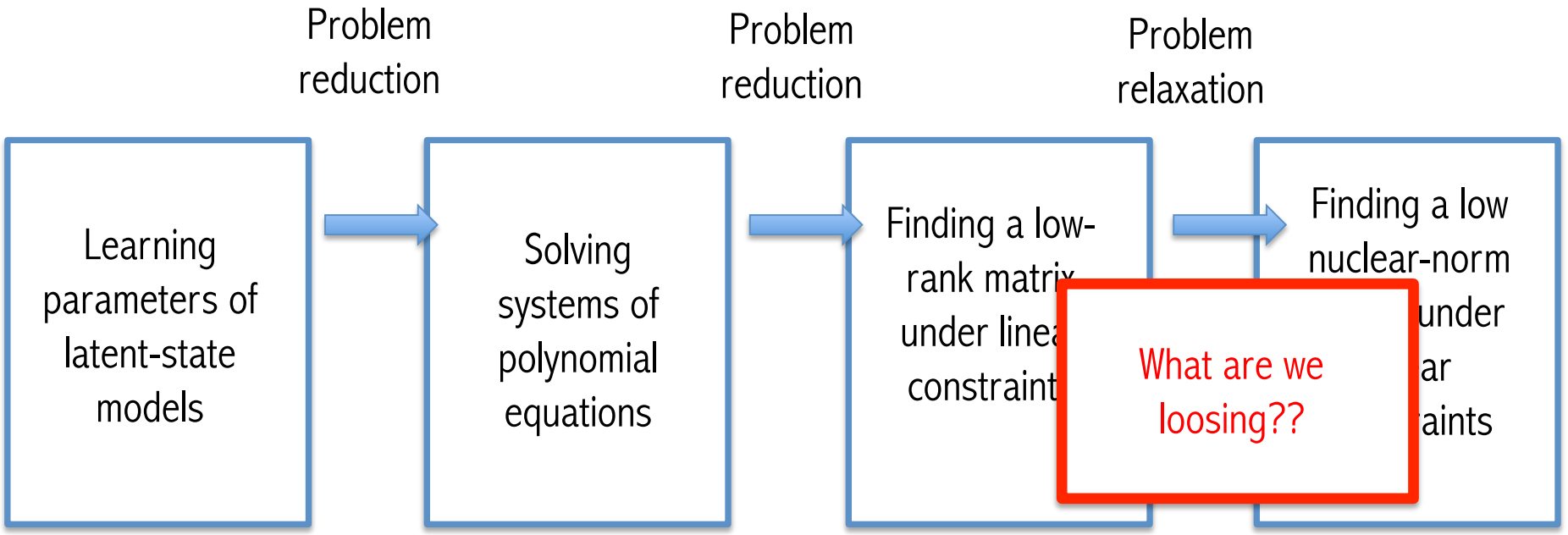

Take-home message: Fundamental ideas behind spectral learning have a wide range of applicability for structured prediction

Thanks!